THE TWENTY-FIFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-fifth annual meeting of the Society was held in Chicago on Friday and Saturday, December 27–28, 1918. The attendance included the following sixty members:

Professor R. C. Archibald, Professor C. S. Atchison, Professor R. P. Baker, Captain A. A. Bennett, Professor H. F. Blichfeldt, Dr. H. Blumberg, Professor J. W. Bradshaw, Professor H. T. Burgess, Professor W. D. Cairns, Professor J. A. Caparo, Professor R. D. Carmichael, Professor H. E. Cobb, Professor A. B. Coble, Professor C. E. Comstock, Mr. H. W. Curjel, Professor D. R. Curtiss, Professor L. E. Dickson, Professor E. L. Dodd, Mr. E. B. Escott, Professor W. B. Ford, Professor Tomlinson Fort, Professor O. E. Glenn, Miss A. B. Gould, Dr. J. O. Hassler, Professor Olive C. Hazlett, Professor T. H. Hildebrandt, Professor L. C. Karpinski, Professor A. M. Kenyon, Professor K. Laves, Professor S. Lefschetz, Professor A. C. Lunn, Professor Gertrude I. McCain, Professor M. McNeill, Professor W. D. MacMillan, Professor Bessie I. Miller, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Mr. E. E. Moots, Professor E. J. Moulton, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaught, Professor G. W. Smith, Professor E. B. Stouffer, Professor E. J. Townsend, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor L. G. Weld, Professor E. J. Wilczynski, Dr. C. E. Wilder, Professor F. B. Wiley, Professor R. E. Wilson, Professor C. H. Yeaton.

At the first session, held Friday afternoon, with Professor Curtiss in the chair, President L. E. Dickson presented his retiring address on "Mathematics in War Perspective." Following this address there was a joint session of the Society with the Mathematical Association of America at which President Dickson presided. The programme at this session was devoted to mathematical problems in connection with the war. The following papers were presented:

(1) Captain A. A. BENNETT: "Some mathematical features of ballistics."

(2) Professor KURT LAVES: "How the map problem was met in the war."

(3) Miss ALICE B. GOULD: "Notes concerning recent books on navigation."

(4) Professor H. L. RIETZ: "Statistics methods for preparation for war department service."

(5) Major W. D. MACMILLAN: "Ordnance problems."

(6) Lieutenant P. L. ALGER: "Practical exterior ballistics."

(7) Professor W. H. ROEVER: "The effect of the earth's rotation and curvature on the path of a projectile."

(8) Professor H. F. BLICHFELDT: "On low velocity high angle fire."

Brief accounts of these papers will be published in the report of the meeting by the Secretary of the Association in the *American Mathematical Monthly*.

On Friday evening the two organizations held a joint dinner at the Quadrangle Club, attended by sixty-eight members and friends.

At the session of the Society on Saturday, President Dickson presided in the morning and Professor Curtiss in the afternoon. The Council, which met Saturday morning, made the following announcements: There was one election to membership in the Society, Mr. V. S. Mallory, graduate student at Columbia University; and eight new applications for membership were received. The Treasurer's report was accepted, having been examined by the auditing committee. It shows a balance of $9,965.28, including the life membership fund of $6,843.46. On recommendation of the committee appointed to arrange for a summer meeting and colloquium at Chicago in 1919, it was voted to postpone that meeting for one year. The usual summer meeting, without colloquium, will be held in 1919 at a place to be determined.

At the annual election which closed Saturday noon, there were one hundred and fifty-nine votes cast. The following officers and other members of the Council were chosen:

President,	Professor FRANK MORLEY.
Vice-Presidents,	Professor G. D. BIRKHOFF, Professor FLORIAN CAJORI.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1921,

Professor H. E. HAWKES, Professor A. C. LUNN,
Professor W. A. HURWITZ, Professor C. N. MOORE.

The following papers were presented at the Saturday sessions:

(1) Professor W. B. FORD, "The sum of any series expressed as a definite integral with application to analytic continuation."

(2) Professor HARRIS HANCOCK: "On the foundations of the elliptic functions."

(3) Professor E. L. DODD: "A comparison of the median and the arithmetic mean of measurements for various laws of error."

(4) Professor DANIEL BUCHANAN: "Asymptotic orbits near the straight line equilibrium points in the problem of three finite bodies."

(5) Professor G. A. MILLER: "Groups containing a small number of sets of conjugate operators."

(6) Professor OLIVE C. HAZLETT: "A theorem on modular covariants."

(7) Mr. WAYNE SENSENIG: "Concerning the covariant theory of involutions of conics."

(8) Professor C. H. SISAM: "On surfaces containing two pencils of cubic curves."

(9) Professor E. B. STOUFFER: "On singular ruled surfaces in S_5."

(10) Dr. HENRY BLUMBERG: "A property of linear point sets."

(11) Professor E. J. WILCZYNSKI: "An application of line geometry to the theory of functions" (preliminary communication).

(12) Professor C. N. MOORE: "On the integro-derivative and some of its applications."

(13) Dr. I. A. BARNETT: "Differential equations with a continuous infinitude of variables."

(14) Dr. BARNETT: "Linear integro-differential equations with constant kernels."

(15) Dr. A. R. SCHWEITZER: "On iterative functional equations of the distributive type."

(16) Professor R. P. BAKER: "Valuation of railroad ties."

(17) Professor BAKER: "Asymptotic forms in probability."

(18) Professor HARRIS HANCOCK: "Rational and integral expressions for the roots of the biquadratic."

(19) Professor ARNOLD EMCH: "On a certain class of rational ruled surfaces."

(20) Dr. W. G. SIMON: "Two formulas of interpolation in two variables."

(21) Professor J. B. SHAW: "On triply orthogonal congruences."

(22) Professor S. LEFSCHETZ: "Real folds of abelian varieties."

(23) Dr. T. H. GRONWALL: "On surfaces of constant curvature with an equation of the form $u(x) + v(y) + w(z) = 0$."

(24) Dr. GRONWALL: "A theorem on power series with application to conformal mapping."

(25) Dr. GRONWALL: "Equipotential minimal surfaces."

(26) Dr. GRONWALL: "On Kummer's series."

(27) Professor P. J. DANIELL: "Integrals in an infinite number of dimensions."

(28) Professor C. N. MOORE: "Generalized limits in general analysis."

(29) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group (fourth paper)."

Mr. Sensenig was introduced by Professor Glenn, and Dr. Barnett by Professor Wilczynski. Professor Hancock's second paper was read by Professor C. N. Moore. In the absence of the authors, the papers by Professor Buchanan, Mr. Sensenig, Professor Sisam, Professor Emch, Dr. Simon, Dr. Gronwall, Professor Daniell, and the first paper by Professor Hancock were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By a method based on the calculus of residues, Professor Ford obtains an expression, in the form of a double improper integral, for the sum of any (convergent) series. When applied in particular to a power series, this expression preserves

a meaning (in general) for values of the variable lying outside the circle of convergence and thus furnishes the analytic continuation of the function defined by the series. The paper is related to a former paper of the author published in *Liouville's Journal* in 1903.

2. The differential equations through which sn u, cn u, dn u or $\wp u$, $\sqrt{\wp u - e_1}$, $\sqrt{\wp u - e_2}$, $\sqrt{\wp u - e_3}$ are defined may be put in the form of the differences of two logarithmic expressions. When integrated it may be shown that each of these expressions is a uniformly convergent power series. Thus in a direct and simple manner, without the introduction of auxiliary functions such as the *Al*-functions introduced in this connection by Weierstrass, Professor Hancock shows that the above functions become quotients of power series which are in fact theta functions.

3. Professor Dodd finds that with any symmetric law of error, the reliability of the median of measurements increases approximately as the square root of the number of measurements, whereas the reliability of the arithmetic mean may remain constant, as in the case of the law

$$\frac{c}{\pi}\frac{dx}{c^2 + x^2},$$

or even decrease. With certain non-Gaussian distributions, the median has a better chance than the arithmetic mean of lying close to the true value.

The so-called probability curve is at best only a good first approximation,—Karl Pearson's curves often give a much better fit. Instead of using a Chauvenet criterion to throw away measurements not conforming to this first approximation, it would appear better to ascertain more closely the law of error for a given distribution—if the number of measurements is large enough—and to base upon this a choice of arithmetic mean, median, or other function of the measurements.

4. It was shown by Lagrange that it is possible to start three finite bodies in such a way that they will describe similar ellipses. There are two configurations of equilibrium which the three bodies will maintain, viz., they are either col-

linear or lie at the vertices of an equilateral triangle. Professor Buchanan determines orbits which are asymptotic, in the sense of Poincaré, to the particular Lagrangian orbits which are described when the three bodies are collinear and move in complanar circles.

5. The object of Professor Miller's paper is to establish a few general theorems relating to the possible number of sets of conjugate operators in a group of a given order and to determine all the possible groups having a small number of complete sets of conjugate operators. When all the operators of a finite group G appear in k sets of conjugates the order of G cannot exceed the kth term diminished by unity in the series

$$2, 3, 7, 43, 1807, \ldots$$

where each term after the first is obtained by multiplying together all the preceding terms and increasing by unity the product thus obtained. In particular, a necessary and sufficient condition that a discontinuous group is of finite order is that its operators appear in a finite number of sets of conjugates under the group.

There is one and only one simple group of composite order which has the property that all of its operators appear in exactly five complete sets of conjugates, viz., the icosohedral group. The octic group, the quaternion group and the metacyclic group of order 20 are the only non-abelian groups whose operators appear in five complete sets of conjugates and whose commutator quotient group is of order 4. If all the operators besides the identity of a non-abelian group belong to two sets of conjugates under its group of isomorphisms, the order of this non-abelian group is either of the form p^a or of the form $p^a - q$, p and q being prime numbers. If a non-abelian group of order p^a has the property that all of its operators besides the identity appear in three sets of conjugates under its group of isomorphisms, then its commutator subgroup is of order $p^{a/2}$.

6. Professor Hazlett's paper answers a question which Dr. Sanderson raised in her thesis,* but did not answer. In her paper the fundamental theorem on the relation between formal and modular invariants for the $GF[p^n]$ enables us to construct modular covariants of a system S of binary forms in

* *Transactions Amer. Math. Society*, vol. 14 (1913), p. 490.

x and y from modular invariants of the same system S in the variables ξ and η and an additional linear form in ξ and η with coefficients y and $-x$. This is closely analogous to the situation in the algebraic theory of invariants. In the latter theory the converse is known to be true—that is, we get *all* covariants in this manner. In the case of modular invariants, however, we do not obtain all covariants, for the universal covariant $L = xy^{p^n} - yx^{p^n}$ can not be obtained as an invariant of a linear form as it vanishes whenever x and y are in the $GF[p^n]$, as we suppose the coefficients of our forms to be. In the paper referred to above, the question was raised as to whether all the covariants of a system S can be expressed as functions of L and the modular invariants of S and an additional linear form. The present paper proves that such is the case, and then, by induction, extends this theorem to a system of forms in n sets of binary variables which are transformed cogrediently.

7. Mr. Sensenig's paper contains the development and tabulation of the simultaneous concomitant set of the pencil of conics $a_x^2 + kb_x^2$, taken with the harmonic conic $(\alpha\beta x)^2$, the twenty forms being expressed as polynomials in k with coefficients which are rational expressions in the twenty forms in the known system of a_x^2 with b_x^2. The set is given in general and also in normal form. The methods are those of generalized ternary transvection, and reduction by symbolical identities. There is derived in the paper, also, the complete system of $a_x^2 + kb_x^2$ taken with a third general conic c_x^2. These forms are reduced in terms of sixty-one concomitants of a_x^2, b_x^2, c_x^2 from the system of three conics first derived by H. F. Baker.

The paper is to appear in the *American Journal of Mathematics*.

8. In a previous paper, Professor Sisam has determined the types of surfaces generated by an algebraic system of cubic curves which do not constitute a pencil. In the present paper he classifies and discusses the surfaces which contain two pencils of cubics, so that every type of algebraic surface with two or more cubic curves through each point is now determined. Each surface is represented birationally on as simple a surface as possible, the complete linear system to which the curves

corresponding to the plane or hyperplane sections belong is determined and a number of characteristic properties of the various types of surfaces are pointed out.

9. By means of a system of three ordinary linear homogeneous differential equations of the second order, Professor Stouffer studies the properties of singular ruled surfaces in space of five dimensions, including ruled surfaces in space of four dimensions. Only surfaces developable in the ordinary sense are excluded from the theory. The geometrical properties of various curves on the surface are shown, the transversal surface is obtained, and some of the relations between a surface and its transversal surface are studied.

10. Let S be any linear point set whatsoever. Let P be a point of S, i an interval enclosing P, l_i the length of i, and $m_e(i, S)$ the exterior measure (Lebesgue) of the subset of S in i. Defining the "exterior épaisseur" of S at P as the limit (if it exists) of $m_e(i, S)/l_i$ as l_i approaches 0, Dr. Blumberg proves that, except for a set of measure 0, the exterior épaisseur of S exists and is equal to 1 at every point P of S. As a consequence, every set may be represented as the sum of two sets, the first being of measure 0 and the second of exterior épaisseur 1 at every one of its points.

11. Let $w = u + iv = f(x + iy) = f(z)$ be a function of the complex variable z. Place the w plane in a position parallel to the z plane with the u and v axes parallel to the x and y axes respectively. The lines which join a point (x, y) of the z plane to the corresponding point (u, v) of the w plane form a congruence. This geometric image of a function $w = f(z)$ has been proposed by several authors, but so far no general results seem to have been obtained by following out this point of view. Professor Wilczynski has found that the congruence obtained in this way has very remarkable properties. Its focal surface is composed of two imaginary cones whose vertices are the circular points of the z plane, and whose generators correspond to the null lines of the z plane. The harmonic conjugate of the z plane with respect to the two focal cones is a real surface with real asymptotic curves whose properties have not yet been completely determined.

12. In this paper Professor C. N. Moore introduces a new generalized limit to replace the derivative of a function at points where the derivative in the ordinary sense and the so-called generalized derivative fail to exist. This limit is designated as the first integro-derivative of the rth order at the point in question and is given by the formula

$$\lim_{u\to 0}\frac{1}{u}\int_0^u dt_r\left(\frac{1}{t_r}\int_0^{t_r} dt_{r-1}\right.$$
$$\left.\times\left(\cdots\left(\frac{1}{t_2}\int_0^{t_2}\frac{f(x+t_1)-f(x-t_1)}{2t_1}dt_1\right)\cdots\right)\right).$$

The utility of this conception is illustrated by means of a theorem which establishes the summability $(C, k > r+1)$ of the derived Fourier's series to the value of the first integro-derivative of the rth order at points where this limit exists.

13. Dr. Barnett discusses the equation

$$\frac{\partial u(\xi, z)}{\partial z} = f(\xi, z, u(\xi')),$$

where ξ, ξ' are real variables in the range $(0, 1)$, z is a real variable on $|z - z_0| \leqq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max|u - u_0| \leqq \beta$, and $f(\xi, z, u)$ is a functional yielding for each (ξ, z, u) of the above ranges a real number. In the first part of the paper hypotheses are stated which are sufficient to insure the existence of a unique solution reducing for $z = z_0$ to an arbitrary continuous function $u_0(\xi)$ and having certain desirable continuity properties.

The solution is then considered as also depending on the initial conditions and theorems concerning the continuity and differentiability are proved. It is finally shown that there exists a solution to the equation

$$\frac{\partial g(z, u)}{\partial z} + \int_0^1 f(\xi, z, u)d_\xi g'(\xi, z, u) = 0,$$

where $g(z, u)$ is the functional to be determined, $f(\xi, z, u)$ is a known functional and $g'(\xi, z, u)$ is a functional related to the differential of $g(z, u)$ defined in the paper. The integration is taken in the sense of Stieltjes.

14. The most general solution of the integro-differential equation

$$\frac{\partial u(\xi, z)}{\partial z} = \int_0^1 k(\xi, \eta) u(\eta, z) dz \tag{1}$$

has already been given by Volterra in the form

$$u(\xi, z) = w(\xi) + \int_0^1 \sum_{\rho=0}^{\infty} \frac{z^\rho}{\rho!} k_\rho(\xi, \eta) w(\eta) d\eta,$$

where the $w(\xi)$ is an arbitrary continuous function on $(0, 1)$ and the functions k_ρ are the iterated kernels of k. Dr. Barnett shows that when the kernel k is symmetric or skew-symmetric, the solution takes on a simple form, analogous to the exponential form of the solution of a finite system of linear differential equations with constant coefficients.

The theorem for symmetric kernels is as follows:

Through the element (z_0, u_0) where z_0 is a real number and u_0 a real continuous function of ξ on $(0, 1)$ there exists one and but one solution of equation (1), viz.,

$$u(\xi, z) = u_0(\xi) + \sum_{\rho=1}^{\infty} c_\rho \varphi_\rho(\xi)(e^{(z-z_0)/\lambda_\rho} - 1)$$

where the φ_ρ are a complete normal orthogonal set of characteristic functions of k, λ_ρ the corresponding characteristic numbers and the c_ρ the Fourier coefficients of $u_0(\xi)$ with respect to the φ_ρ, *i. e.*, $c_\rho = \int_0^1 \varphi_\rho(\eta) u_0(\eta) d\eta$.

A corresponding theorem for a special type of non-homogeneous equation is also discussed.

15. Dr. Schweitzer defines the "quasi-distributive" equations of order $k \geqq 1$; an important instance is

$$\begin{aligned} &\phi[\phi(x_1, \cdots, x_{n+1}), t_1, \cdots, t_n] \\ &\quad = \phi[x_1, \phi(x_2, t_1, \cdots, t_n), \cdots, \phi(x_{n+1}, t_1, \cdots, t_n)] \end{aligned} \tag{1}$$

where $n = 1, 2, 3$, etc. For $n = 1$ this relation is the associative relation. In the domain of finite or infinite abstract groups* relation (1) is always solvable. Analytically, the

* Cf. abstract, BULLETIN, Nov., 1918, pp. 58, 59. In this abstract for $k = 3$ read $k = 2$; for $k = 2$ read $k = 1$; for interpolation read interpretation.

equation (1) has the solution

$$\phi(x_1, x_2, \cdots, x_{n+1}) = \psi^{-1}[\psi(x_1) + \psi(x_2) - \Theta\{\psi(x_2) - \psi(x_3), \cdots, \psi(x_2) - \psi(x_{n+1})\}]$$

where Θ is an "arbitrary" function of $n-1$ variables and ψ is an "arbitrary" function with an inverse. For $x_1 =$ constant in (1) this solution suggests immediately a particular solution of a certain properly distributive equation on two functions.

The equation (1) in connection with the equations

$$\begin{aligned} f[\phi(x_1, t_1, t_2, \cdots, t_n), t_1, \cdots, t_n] &= x_1, \\ \phi[f(x_1, t_1, t_2, \cdots, t_n), t_1, \cdots, t_n] &= x_1, \end{aligned} \tag{2}$$

yields, as an eliminant in the function f, an "identity" quasi-transitive functional equation (and conversely). This result is generalized for a finite group of quasi-transitive functional equations and thus leads to the concept of a finite group of quasi-distributive functional equations and, subsequen ly, to a finite group of properly distributive equations on two functions.

16. Professor Baker gives the formula

$$\Sigma_s[\{\Sigma_v \cdot \text{Coeff. } x^s m(\Sigma_r p_r x^{l_r})^v\} \cdot \Sigma_r p_r(l_r - X + s)],$$

divided by a denominator obtained from this by replacing the last Σ_r by 1, for the most probable unexpired life of a railroad tie. Here p_r is the probability of a tie having a life l_r; Σ_r runs over all possible lives, Σ_v from $v = [X/M]$ to $v = [X/m]$ where X is the age of the road and M, m are the maximum and minimum life of a tie. Σ_s runs from $s = X - M$ to X including the first term if the result is desired immediately before the annual relaying and the last term if the result refers to a time just after this. The asymptotic form is obtained by omitting the Σ_v and replacing it by 1. This holds for all positive distributions (p_r). A large enough X to use this form has not been reached; in actual cases the result oscillates about the asymptotic amount, which is greater than the text rule of "half the average life" for a uniform distribution and less than this for a skew binomial.

Some results are as follows: taking the mean average of the

two Σ_s's and $m = 5$, $M = 15$, $X = 54$ to 60 inclusive, and for the two cases of uniform distribution and skew binomial $(2 + x)^{10}$, we have

	M. A.	Asym.	Text	Oscillation
U. D.	5.266	5.25	5	1.8%
S. B.	4.037	4.032	4.167	2.2%

An oscillation of 1% is approximately $10,000,000 for the roads of the United States.

17. Professor Baker considers the integral equations

$$kf(bx) = \int_{-\infty}^{\infty} f(u)f(x-u)du \tag{1}$$

and

$$kf(bx) = \int_{0}^{x} f(u)f(x-u)du. \tag{2}$$

In volume 76 of the *Mathematische Annalen* Runge and Pólya have discussed the solution of

$$f(x) = \int_{-\infty}^{\infty} \phi(u)\phi(x-u)du.$$

Runge's method adds nothing to the discussion, but Pólya's makes the general solution depend on the functional equation

$$[\phi(s)]^2 = c\phi(as).$$

The resulting asymptotic forms include all of De Forest's based on polynomials, and also others, the simplest being $1/(1+x^2)$. This has an asymptotic neighborhood among continuous functions, and among double ended series, but not among polynomials. The equation (2) has no proper solution.

18. By making use of the associated realms of rationality, Professor Hancock is able to show that any root of a biquadratic may be expressed rationally and also integrally through any other root with coefficients that are rational and integral functions of the roots of the reducing cubic.

For example if $\sigma = \sqrt{p} + \sqrt{q} + \sqrt{r}$ and $\tau = \sqrt{p} - \sqrt{q} - \sqrt{r}$

the rather remarkable relation is

$$\tau = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3,$$

where the A's are integral functions of the second degree in p.

19. It is well known that algebraic ruled surfaces may be generated by the lines joining corresponding points of an (α, β)-correspondence between the points of two algebraic curves in space. Professor Emch considers a large class of ruled surfaces defined by certain cinematic properties, which are equivalent to such a correspondence between the points of a straight line and a circle.

The parametric equations of these surfaces in cartesian coordinates are

$$x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad z = (\rho - a) \cot p\theta/q,$$

where ρ and θ are the parameters, and p and q two integers which are relatively prime. It is shown that these surfaces are rational so that every plane section is a rational curve. They may be divided into two sub-classes of bifacial and unifacial surfaces according as q is even or odd.

When q is odd, the order of the surface is $2(p + q)$. In this case the directrix circle and the directrix line are respectively q-fold and $2p$-fold curves of the surface. Besides these the surface has $\frac{1}{2}(q + 1)$ real double curves of order $4p$ or $2q$, according as $2p \gtrless q$, and p real double lines.

When $q = 2s$ is even, then the order of the surface is $p + q$. The directrix circle and line are s-fold and p-fold respectively. There are moreover $\frac{1}{2}(s + 1)$ or $\frac{1}{2}s$ real double curves, according as s is odd or even; they are of order $2p$ or q, according as $2p \gtrless q$, except when s is odd; then there are $\frac{1}{2}(s - 1)$ curves of order $2p$ and one curve of order p.

The system of points of intersection of the double curves with a plane through the directrix line may be arranged according to certain cyclic groups of various orders.

These surfaces have the remarkable property that, in certain sets, they are applicable upon each other, and their intersections with a torus lead to all so-called cyclo-harmonic curves.

It is also shown how Moebius' unifacial and bifacial bands of all orders may be cut out from these surfaces.

20. In a recent paper (*Annals of Mathematics*, volume 19,

number 4) Dr. Simon gave a formula of polynomial interpolation for a continuous function of a single real variable. The formula

$$\sigma_n(x, y) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(x_i, y_j)[1 - (x_i - x)^2 - (y_j - y)^2]^n}{2\sum_{i=0}^{n}\sum_{j=0}^{n}[1 - x_i^2 - y_j^2]^n},$$

where

$$x_i = \frac{i}{\sqrt{2}n}, \quad y_j = \frac{j}{\sqrt{2}n},$$

and the function $f(x, y)$ is defined and continuous when

$$0 < a \leqq x \leqq b \leqq \frac{1}{\sqrt{2}}, \quad 0 < c \leqq y \leqq d \leqq \frac{1}{\sqrt{2}},$$

is proposed in the present paper, and is the immediate extentension of the earlier one to a continuous function of two real variables. Furthermore, it is shown that when the function which is being approximated satisfies a Lipschitz condition, the order of approximation is $1/\sqrt{n}$.

A trigonometric formula of approximation is also proposed for a continuous function of two real variables having the period 2π. This formula

$$\Sigma_n(x, y) = \frac{\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1} f(x_i, y_j)\left[\cos\frac{x_i - x}{2}\cos\frac{y_j - y}{2}\right]^{2n}}{2\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\cos\frac{x_i}{2}\cos\frac{y_j}{2}\right]^{2n}},$$

where the x_i and y_j are two sets of values such that $x_{i+1} - x_i = \pi/n$, $y_{j+1} - y_j = \pi/n$, is the extension of one given by Kryloff (*Bulletin des Sciences Mathématiques*, volume 41, October, 1917). Here too the order of approximation is $1/\sqrt{n}$ when the function satisfies a Lipschitz condition.

21. A congruence of curves is exemplified in the lines of a field of force, or velocity. In many questions it becomes necessary to study the rate of variation around a point of the unit vector which defines the congruence. Also in relation to the unit vector α which is tangent to the vector line (line of the congruence) at the point are other perpendicular unit vec-

tors, such as the principal normal and the binormal and others. Professor Shaw's paper is a study of the properties of such triply orthogonal sets of unit vectors. A linear vector function θ is defined which gives, for a differential displacement of the point ηds, the instantaneous axis and amount of rotation of the triply orthogonal system α, β, γ, that is $\theta(\eta)\cdot ds$. It turns out that θ is symmetric in α, β, γ and every theorem is thus true at once for each unit vector or its proper congruence. Also the curl and the divergence of each can be stated in terms of θ. The lines of equi-directed unit-tangents of the congruence are found in terms of θ. The invariant axes of θ define sets of straight lines along which the axis of rotation of the trihedral is in the direction of the displacement. Other properties of curvatures and torsions are discussed. (Cf. Rogers, *Proceedings of the Royal Irish Academy*, volume 29, section A, No. 6 (1912), pages 92–117.) Quaternion methods are used.

22. The object of Professor Lefschetz's note is to show that an abelian variety V_p of genus p and rank one can have any number of real folds represented by 2^s, $s \geqq p$. These folds form equivalent p dimensional cycles of V_p and their indices of connection are easily determined. There are Jacobi varieties of each of the p types obtained and their consideration yields a very simple proof of the theorem due to Harnack, according to which a curve of genus p can have at most $p+1$ real branches. As a particular case of the above results, there are hyperelliptic surfaces of arbitrary divisor and rank one, with one, two or four real folds forming as many equivalent two dimensional cycles and each reducible by deformation to a ring.

23. In this paper, Dr. Gronwall shows that the determination of the surfaces indicated in the title reduces to the solution of the equation

$$f'(u)g'(v)h(w) + g'(v)h'(w)f(u) + h'(w)f'(u)g(v) = [f(u) + g(v) + h(w)]^2$$

under the condition $u + v + w = 0$. By function theoretic means, involving an extensive use of Hadamard's theory of entire functions of finite genus, it is shown that this functional equation has only one solution, which corresponds to the surfaces of revolution of constant curvature.

24. Dr. Gronwall proves the following theorem: When the power series $w(z) = \sum_0^\infty a_n z^n$ converges for $|z| < 1$, and $\sum_0^\infty n|a_n|^2$ converges, and we write $z_1 = 1 - \rho e^{\theta i}$, $z_2 = 1 - \rho e^{\theta' i}$, where $\rho > 0$ and θ and θ' vary with ρ subject only to the conditions

$$-\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{2} + \epsilon \leqq \theta' \leqq \frac{\pi}{2} - \epsilon,$$

then

$$w(z_1) - w(z_2) \to 0 \text{ as } \rho \to 0$$

uniformly in respect to θ and θ'. This theorem is useful in the study of the behavior of a conformal mapping function on the boundary of the mapped region.

25. The determination of all harmonic functions $V(x, y, z)$ such that $V(x, y, z) = \text{const.}$ defines a family of minimal surfaces is of a certain importance in hydrodynamics. In the present paper, Dr. Gronwall shows that beyond the known case $V = z - a \arctan y/x$ (and those derived from it by a change of coordinate axes), no such functions exist.

26. In this paper, Dr. Gronwall calls attention to the fact that many of the published proofs of the expansion

$$\log \Gamma(s) + \tfrac{1}{2} \log \frac{\sin \pi s}{\pi} + (s - \tfrac{1}{2})(\log 2\pi - c) = \sum_1^\infty \frac{\log n}{n\pi} \sin 2n\pi s$$

are deficient, and gives two new and elementary proofs.

27. In the *Annals of Mathematics*, June, 1918, Professor Daniell published a paper on "A general form of integral." By using the methods of that paper, he is now able to define integrals in an infinite number of dimensions, that is, not merely line integrals in an infinitely dimensional space. In a second part, the author defines a function which is of limited variation in an infinite number of dimensions and thereby is able to define a general Stieltjes integral.

28. Methods analogous to the methods for summing divergent series and integrals may be employed for defining derivatives of functions at points where the ordinary derivative fails to exist. This suggests building up a general theory which might be termed the theory of generalized limits in general analysis. In Professor C. N. Moore's paper a general theorem in this theory is established which includes as special cases the Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder methods for summing divergent series, the corresponding theorem for integrals due to Landau, and a third theorem with regard to the equivalence of the integro-derivatives of the Cesàro and Hölder type.

29. In a former paper Dr. Schweitzer has shown that a certain "quasi-distributive" property is satisfied by the elements of an abstract group, finite or infinite. This suggests that properly distributive functional equations have solutions in the domain of abstract groups. In verification, Dr. Schweitzer shows that the relation

$$\zeta[\tau(x_1, x_2, \cdots, x_{n+1})t_1, t_2, \cdots, t_m] = \tau[\zeta(x_1, t_1, \cdots, t_m), \cdots, \zeta(x_{n+1}, t_1, \cdots, t_m)] \quad m \geqq 1,\ n \geqq 1$$

has the solution (among others)

$$\tau(x_1, x_2, \cdots, x_{n+1}) = x_1 x_2^{-1} x_1 \cdots x_{n+1}^{-1} x_1,$$

$$\zeta(x_1, t_1, \cdots, t_m) = x_1 t_1^{p_{11}} t_2^{p_{12}} t_1^{p_{12}} \cdots t_m^{p_{1m}} t_1^{p_{1m}},$$

where the p's are arbitrary integers not zero. Further, the equation on one function,

$$\tau\{x, \tau(y, z)\} = \tau\{\tau(x, y), \tau(x, z)\},$$

is satisfied by $\tau(x, y) = xy^{-1}x$. Finally, the author gives the following set of postulates (apart from closure) for an abstract group based on distributiveness:

1. $\zeta[\tau\{x, \zeta(x, y)\}, \zeta(y, z)] = \zeta(x, z)$.
2. $\zeta[\tau(x, y), z] = \tau[\zeta(x, z), \zeta(y, z)]$.
3. $\tau[x, \zeta(x, x)] = \tau[y, \zeta(y, y)]$.
4. $\tau[\zeta(x, y), x] = \zeta[\zeta(x, y), y]$.
5. $\tau[x, \tau(x, y)] = y$.

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Acting Secretary.

APPLICATIONS OF THE THEORY OF SUMMABILITY TO DEVELOPMENTS IN ORTHOGONAL FUNCTIONS.

BY PROFESSOR CHARLES N. MOORE.

(Read at the Chicago Symposium of the American Mathematical Society April 12, 1918.)

THE very considerable body of literature which may be described by the above title belongs almost entirely to the present century. Its extent is only roughly indicated by the bibliography at the end of the paper, which makes no pretensions to being complete. The type of series considered here constitutes one of the three most important classes of series to which the theory of summability has been applied, the other two being power series and Dirichlet's series. A noteworthy feature of the applications with which we shall be concerned is found in their usefulness in an important branch of applied mathematics, namely the theory of the flow of heat and electricity.

§1. *The Summability of Fourier's Series.*

The first writer to deal with the topic of this section was Fejér. In his fundamental paper of 1903 [5]* he established among other results the summability $(C1)$ of the Fourier development of an arbitrary function satisfying very wide conditions, at all points where the function is continuous or has a finite jump. We shall give a proof of this theorem, under somewhat modified conditions, which is substantially the same as Fejér's proof.

The Fourier development of $f(x)$ may be written in the form

$$(1) \qquad \frac{1}{2\pi}\int_a^{2\pi+a} f(\theta)d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_a^{2\pi+a} f(\theta)\cos n(\theta - x)d\theta,$$

it being assumed that $f(x)$ is periodic of period 2π. We have for the sum of the first n terms of (1)

$$(2) \qquad s_n(x) = \frac{1}{\pi}\int_a^{2\pi+a} f(\theta)\frac{\cos(n-1)(\theta - x) - \cos n(\theta - x)}{2(1 - \cos(\theta - x))}d\theta.$$

* The numbers in brackets refer to the bibliographical list at the end of the paper.

Hence for the arithmetic mean of the first n sums, we have

$$\frac{S_n(x)}{n} = \frac{1}{2n\pi}\int_a^{2\pi+a} f(\theta)\frac{\sin^2\frac{n(\theta-x)}{2}}{\sin^2\frac{\theta-x}{2}}\,d\theta. \tag{3}$$

If now we set $t = \frac{1}{2}(\theta - x)$ and then take $\alpha = x - \pi$, we obtain

$$\frac{S_n(x)}{n} = \frac{1}{n\pi}\int_{-(\pi/2)}^{\pi/2} f(x+2t)\frac{\sin^2 nt}{\sin^2 t}\,dt \tag{4}$$

$$= \frac{1}{n\pi}\int_0^{\pi/2}\{f(x+2t)+f(x-2t)\}\frac{\sin^2 nt}{\sin^2 t}\,dt.$$

We are now ready for the proof of Fejér's theorem. We begin by establishing two lemmas.

LEMMA 1. *For any positive integer n, we have*

$$\frac{1}{n\pi}\int_0^{\pi/2}\frac{\sin^2 nt}{\sin^2 t}\,dt = \tfrac{1}{2}. \tag{5}$$

If we set

$$\sigma_n(2t) = \tfrac{1}{2} + \cos 2t + \cos 4t + \cdots + \cos 2(n-1)t \tag{6}$$

$$= \frac{\cos 2(n-1)t - \cos 2nt}{2(1-\cos 2t)},$$

we have

$$\frac{\Sigma_n(2t)}{n} = \frac{\sigma_1(2t)+\sigma_2(2t)+\cdots+\sigma_n(2t)}{n} = \frac{1}{2n}\cdot\frac{\sin^2 nt}{\sin^2 t}. \tag{7}$$

But from (6)

$$\frac{2}{\pi}\int_0^{\pi/2}\sigma_n(2t)dt = \tfrac{1}{2} \qquad (n = 1, 2, 3, \cdots).$$

From this relationship and (7) the identity (5) readily follows.

LEMMA 2. *If $\varphi(t)$ is integrable (Lebesgue) in the interval $(0 \leqq x \leqq c \leqq \frac{1}{2}\pi)$ and furthermore $\lim_{t\to 0}\varphi(t) = 0$, then*

$$\lim_{n\to\infty}\frac{1}{n\pi}\int_0^c \varphi(t)\frac{\sin^2 nt}{\sin^2 t}\,dt = 0.$$

Given an arbitrary positive ϵ, we chose $\delta < \epsilon$ and such that

$$|\varphi(t)| < \epsilon \qquad (0 \leqq t \leqq \delta). \tag{8}$$

Then, δ being fixed, we choose m so large that

$$\frac{1}{n\pi \sin^2 \delta} \int_\delta^c |\varphi(t)|\, dt < \frac{\epsilon}{2} \qquad (n \geqq m). \tag{9}$$

From (8), (5), and (9) we obtain

$$\left| \frac{1}{n\pi} \int_0^c \varphi(t) \frac{\sin^2 nt}{\sin^2 t}\, dt \right| \leqq \frac{1}{n\pi} \int_0^\delta |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t}\, dt$$
$$+ \frac{1}{n\pi} \int_\delta^c |\varphi(t)| \frac{\sin^2 nt}{\sin^2 t}\, dt < \frac{\epsilon}{n\pi} \int_0^{\pi/2} \frac{\sin^2 nt}{\sin^2 t}\, dt$$
$$+ \frac{1}{n\pi \sin^2 \delta} \int_\delta^c |\varphi(t)|\, dt < \epsilon \qquad (n \geqq m),$$

which proves our lemma.

FEJÉR'S THEOREM.* *If the function $f(x)$ is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π, the series* (1) *will be summable* (C1) *to the value $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at every point for which this limit exists.*

It is obvious that this theorem can be proved by showing that

$$\lim_{n \to \infty} \left[\frac{S_n(x)}{n} - \tfrac{1}{2}\{f(x+0) + f(x-0)\} \right] = 0,$$

where $S_n(x)$ is defined by (3). In view of (4) and Lemma 1, the expression in brackets may be written in the form

$$\frac{1}{n\pi} \int_0^{\pi/2} \{f(x+2t) + f(x-2t) - f(x+0) - f(x-0)\} \frac{\sin^2 nt}{\sin^2 t}\, dt.$$

It follows at once from Lemma 2 that this last expression approaches zero as n becomes infinite. The theorem is therefore proved.†

* Fejér's original conditions are somewhat different from ours, he having required that $f(x)$ be integrable (Riemann) and become infinite at only a finite number of points. Thus his result and our result overlap. It is easy to modify our argument so as to include both results.

† With slight changes the above argument may be used to establish uniform summability throughout any interval included in an interval of continuity of $f(x)$.

Since the points of discontinuity of a function having a Riemann integral form a set of measure zero, Fejér's theorem for such functions may be stated in the form: *The Fourier development of any function having a Riemann integral is summable* $(C1)$ *to the value of the function at all points except for a set of measure zero.* But Fejér's theorem does not enable us to assert anything of that sort about functions having a Lebesgue integral, since for such functions there may be no points where the limit $\frac{1}{2}[f(x+0)+f(x-0)]$ exists. However, Lebesgue has extended the second form of Fejér's theorem to functions integrable according to his definition [14], in a theorem which we shall now proceed to prove. We begin by establishing two lemmas.

LEMMA 3. *If* $g(t)$ *is positive or zero in the interval* $(0 \leqq t \leqq \frac{1}{2}\pi)$, *has a Lebesgue integral there, and is such that*

$$\lim_{t\to 0}\frac{G(t)}{t}=\lim_{t\to 0}\frac{1}{t}\int_0^t g(t)dt=0, \tag{10}$$

then we shall have

$$\lim_{k\to\infty}\int_0^{\pi/2}\frac{G(t)}{t}\cdot\frac{\sin kt}{t}dt=0. \tag{11}$$

Given an arbitrary positive ϵ, we choose a δ such that

$$0\leqq\frac{G(t)}{t}<\frac{\epsilon}{4\pi}\qquad(0<t\leqq\delta). \tag{12}$$

Then, δ being fixed, we choose m_1, such that

$$\left|\int_\delta^{\pi/2}\frac{G(t)}{t^2}\cdot\sin kt\,dt\right|<\frac{\epsilon}{2}\qquad(k\geqq m_1), \tag{13}$$

which we may do in view of the Riemann-Lebesgue theorem with regard to the limiting values of the Fourier's constants of an integrable function.*

For values of $k>\pi/\delta$ we have, in view of (12),

$$\left|\int_0^{\pi/k}\frac{G(t)}{t}\cdot\frac{\sin kt}{t}dt\right|<\frac{\epsilon}{4\pi}\int_0^{\pi/k}\frac{\sin kt}{t}dt \tag{14}$$

$$=\frac{\epsilon}{4\pi}\int_0^{\pi}\frac{\sin u}{u}du<\frac{\epsilon}{4}.$$

* Cf. Lebesgue, Leçons sur les Séries trigonométriques, p. 61.

Furthermore, on integrating by parts,

$$(15)\quad \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt\, dt = \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} + \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt\, dt - \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt\, dt.$$

But

$$(16)\quad \left| \left[-\frac{1}{k} \cos kt \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} \right| \leqq \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \cdot \frac{G(\pi/k)}{\pi/k},$$

$$(17)\quad \left| \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} \cos kt\, dt \right| < \frac{\epsilon}{2\pi k} \int_{\pi/k}^{\delta} \frac{dt}{t^2} < \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2},$$

$$(18)\quad \left| \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} \cos kt\, dt \right| < \frac{1}{k} \int_{\pi/k}^{\delta} \frac{g(t)}{t^2} dt = \left[\frac{1}{k} \cdot \frac{G(t)}{t^2} \right]_{\pi/k}^{\delta} + \frac{2}{k} \int_{\pi/k}^{\delta} \frac{G(t)}{t^3} dt < \frac{1}{k} \cdot \frac{G(\delta)}{\delta^2} + \frac{1}{\pi} \frac{G(\pi/k)}{\pi/k} + \frac{1}{k} \cdot \frac{\epsilon}{2\pi\delta} + \frac{\epsilon}{2\pi^2}.$$

Combining (15), (16), (17) and (18), and taking into account (12), it is readily seen that we may choose m_2 such that

$$(19)\quad \left| \int_{\pi/k}^{\delta} \frac{G(t)}{t^2} \sin kt\, dt \right| < \frac{\epsilon}{4} \qquad (k \geqq m_2).$$

If we designate by m the greatest of the three quantities m_1, π/δ and m_2, we have from (13), (14) and (19)

$$\left| \int_0^{\pi/2} \frac{G(t)}{t} \cdot \frac{\sin kt}{t} dt \right| < \epsilon \qquad (k \geqq m).$$

Our lemma is therefore proved.

LEMMA 4. *If $f(x)$ is integrable (Lebesgue) in the interval $(a \leqq x \leqq b)$, $|f(x) - f(x_0)|$, where $(a \leqq x_0 \leqq b)$, is for $x = x_0$ the derivative of its indefinite integral, except perhaps at a set of points of measure zero.*

We know from a well known theorem in the theory of Lebesgue integrals that $f(x) - \alpha$, where α is any constant, will, under the conditions of our lemma, be the derivative of its indefinite integral for all points of $(a \leqq x \leqq b)$ except perhaps a set of measure zero. Let this set be represented by $E(\alpha)$, and let E_1 represent the set which is the sum of all the $E(\alpha)$ for all rational values of α. Then E_1 will also be a set of measure zero.

Let x_0 be a value of x in $a \leqq x \leqq b$ which does not belong to E_1, and let β be any irrational number. Given an arbitrary positive ϵ, we can find a rational number α so near to β that

$$(20) \qquad \big|\,|f(x)-\beta| - |f(x)-\alpha|\,\big| \leqq |\alpha-\beta| < \frac{\epsilon}{3},$$

whence

$$(21) \qquad \left|\frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(x)-\beta|\,dx\right| - \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(x)-\alpha|\,dx\right|\right| < \frac{\epsilon}{3}.$$

Since, moreover, the second term of the left hand member of (21) approaches $|f(x)-\alpha|$ as x approaches x_0, we may choose a δ such that

$$(22) \qquad \left|\frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(x)-\alpha|\,dx\right| - |f(x)-\alpha|\right| < \frac{\epsilon}{3} \qquad (|x-x_0|<\delta).$$

Combining (20), (21) and (22), we obtain

$$\left|\frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(x)-\beta|\,dx\right| - |f(x)-\beta|\right| < \epsilon \qquad (|x-x_0|<\delta).$$

Hence $|f(x)-\beta|$ is for $x=x_0$ the derivative of its indefinite integral. Since x_0 was any point not of E_1 and β was any irrational number, it follows that $|f(x)-\gamma|$, where γ is any number, is the derivative of its indefinite integral for every point of $(a \leqq x \leqq b)$ except a set of measure zero. Since for any point x_0 we may choose $\gamma = f(x_0)$, our lemma is proved.

LEBESGUE'S THEOREM. *The Fourier development of a function $f(x)$ that is periodic of period 2π and is integrable (Lebesgue) over any interval of length 2π, is summable (C1) almost everywhere to the value $f(x)$.*

Our theorem may be proved by showing that

$$(23) \qquad \lim_{n\to\infty}\left[\frac{S_n(x)}{n} - f(x)\right] = 0,$$

where $S_n(x)$ is defined by (3), is true almost everywhere. In view of (4) and Lemma 1 the expression in brackets in (23)

may be written in the form

$$(24)\quad \frac{1}{n\pi}\int_0^{\pi/2}[f(x+2t)+f(x-2t)-2f(x)]\frac{\sin^2 nt}{\sin^2 t}\,dt = \frac{1}{n\pi}\int_0^{\pi/2}\varphi_x(t)\frac{\sin^2 nt}{\sin^2 t}dt.$$

We will show that (24) approaches zero as n becomes infinite for all values of x for which $|\varphi_x(t)|$ is for $t=0$ the derivative of its indefinite integral $\Phi_x(t)$. This latter will be the case for all values of x for which both $|f(x+2t)-f(x)|$ and $|f(x-2t)-f(x)|$ are for $t=0$ the derivatives of their indefinite integrals, and in view of Lemma 4 this is true almost everywhere. Thus our theorem will be proved.

We consider then a value of x for which $\Phi_x'(0)=\varphi_x(0)=0$, and hence we have

$$(25)\quad \lim_{t\to 0}\left[\frac{\Phi_x(t)}{t}\right]=0.$$

Integrating by parts in the integral on the right hand side of (24), this expression takes the form

$$(26)\quad \frac{1}{n\pi}\left[\Phi_x(t)\frac{\sin^2 nt}{\sin^2 t}\right]_0^{\pi/2}-\frac{1}{\pi}\int_0^{\pi/2}\frac{\Phi_x(t)}{\sin t}\cdot\frac{\sin 2nt}{\sin t}\,dt + \frac{2}{n\pi}\int_0^{\pi/2}\frac{\Phi_x(t)}{\sin t}\cdot\frac{\sin^2 nt}{\sin^2 t}\,dt.$$

The first term in (26) vanishes at the lower limit in view of (25) and at the upper limit takes on a value which approaches zero as n becomes infinite. From Lemma 2 and (25) the third term is readily seen to approach zero as n becomes infinite. The second term may be replaced by

$$(27)\quad \frac{1}{\pi}\int_0^{\pi/2}\frac{\Phi_x(t)}{t}\cdot\frac{\sin 2nt}{\sin t}\,dt,$$

since the difference between the two approaches zero as n becomes infinite in view of the theorem of Riemann-Lebesgue referred to in the proof of Lemma 3. But it follows from Lemma 3 that (27) approaches zero as n becomes infinite. Hence (26), and therefore (24), has this same property for the value of x we are considering. Thus, as pointed out above, the theorem is proved.

We may also use for the summation of Fourier's series the non-integral orders of summability introduced by Knopp, Marcel Riesz and Chapman. These may be defined as follows: Let

$$(28)\quad s_n = \sum_{m=0}^{m=n} u_m, \qquad A_n^{(k)} = \frac{\Gamma(k+n+1)}{\Gamma(k+1)\Gamma(n+1)},$$

$$S_n^{(k)} = \sum_{p=0}^{p=n} A_{n-p}^{(k-1)} s_p.$$

Then if $\sigma_n^{(k)} = S_n^{(k)}/A_n^{(k)}$ approaches a limit σ as n becomes infinite, we say that the series Σu_n is summable (Ck) with sum σ. We may also define the sum of the series to be

$$(29)\quad \lim_{\omega\to\infty}\left[\sum_{n=0}^{n<\omega}\left(1-\frac{n}{\omega}\right)^k u_n\right]$$

whenever that limit exists. This latter definition has been shown by Riesz to be entirely equivalent to the former one.*

It was shown independently by Riesz [18] and Chapman [3] that the Fourier development of a function $f(x)$ having a Lebesgue integral is summable $(C, k>0)$ at all points at which $\lim_{h\to 0}[\frac{1}{2}\{f(x+h)+f(x-h)\}]$ exists, to the value of that limit. It was shown by G. H. Hardy that the series is summable $(C, k>0)$ to $f(x)$ almost everywhere [13]. Hardy's proof of his theorem is similar in method to a simplified proof of the Riesz-Chapman theorem given by W. H. Young [19] and [20]. Space is lacking to give here the details of these proofs. They depend on properties of the functions

$$C_p(t) = \frac{t^p}{\Gamma(p+1)}\left\{1 - \frac{t^2}{(p+1)(p+2)} + \frac{t^4}{(p+1)(p+2)(p+3)(p+4)} - \cdots\right\}$$

introduced by Young. These functions are generalizations of the sine and cosine, for we have obviously $C_0(t) = \cos t$, $C_1(t) = \sin t$.

§2. *Convergence Factors.*

Convergence factors may be defined as a set of functions of a parameter which, when introduced as factors of the succes-

* Cf. *Comptes Rendus*, June 12, 1911.

sive terms of a series, cause a divergent series to converge,* or a series which is already convergent to converge more rapidly throughout a given range of values of the parameter. In the case of all the convergence factors used in practice, it is further true that each factor approaches unity as the parameter approaches a certain value, and that the function of the parameter defined by the series with the convergence factors approaches a limit as the parameter approaches this same value, this limit being the value of the series for convergent series and a value we find it useful to ascribe to the series in the case of a divergent series.

Thus we see that a set of convergence factors may be used to define the sum of a divergent series. This method of summation goes back to Euler, who frequently arrived at a value for a divergent series Σu_n by setting $\Sigma u_n = \lim_{x \to 1-0} \Sigma u_n x^n$. A simple illustration of this process is exhibited in the case of the series $1 - 1 + 1 - 1 + \cdots$. For this series we have

$$\lim_{x \to 1-0} \Sigma u_n x^n = \lim_{x \to 1-0} (1 - x + x^2 - x^3 + \cdots) = \lim_{x \to 1-0} \frac{1}{1+x} = \tfrac{1}{2}.$$

This value agrees with the value of the series when summed by the mean value process.

From one point of view, practically every method of summing divergent series may be regarded as a convergence factor method. Thus in the summation by first means we seek the limit as n becomes infinite of

$$F(n) = \frac{s_0 + s_1 + \cdots + s_n}{n+1} = u_0 + \left(1 - \frac{1}{n+1}\right) u_1 + \left(1 - \frac{2}{n+1}\right) u_2 + \cdots + \left(1 - \frac{n}{n+1}\right) u_n.$$

The expression on the right hand side may be regarded as the series obtained by introducing the convergence factors

$$f_m\left(\frac{1}{n}\right) = 1 - \frac{m}{n+1} \qquad (m = 0, 1, 2, \cdots, n);$$

$$f_m\left(\frac{1}{n}\right) = 0 \quad (m = n+1, n+2, \cdots)$$

* Some writers have employed the term convergence factor in the case of a set of factors which cause a divergent series to tend toward convergence, i. e., to become summable of a lower index.

into the series Σu_n. Here both the variables of which the convergence factors are functions take on only discrete values and we take the limit of $F(n)$ as n becomes infinite. But the principal characteristic of this type of factors lies in the fact that for any definite value of the parameter they are all zero from a certain point on. In this they differ from the type to which the name convergence factor has ordinarily been applied.

The simplest example of the latter type is to be found in the set of convergence factors mentioned above as used by Euler, namely the set $1, x, x^2, \cdots$. In connection with this set it is interesting to note that the fact we have indicated above with regard to the effect of their introduction into the series $1 - 1 + 1 - 1 + \cdots$, is not an accidental coincidence but a particular case of a general theorem due to Frobenius [8]. This theorem may be stated as follows:

FROBENIUS'S THEOREM: *If the series Σu_n is summable $(C1)$ to the value S, then the series $\Sigma u_n x^n$ will converge for $0 < x < 1$, and the function $F(x)$ which it defines for those values will be such that* $\lim_{x \to 1-0} F(x) = S$.

We shall not give the proof of this theorem, inasmuch as it is a special case of a more general theorem due to Bromwich [1]. This latter theorem includes also a number of other theorems about convergence factors due to various writers. In its most general form it applies to the introduction of convergence factors into series summable (Ck), where k is any positive integer.* We shall deal only with the simplest case where $k = 1$, but for this case we will state the theorem in somewhat more general form than it is given by Bromwich. It requires only slight changes of phraseology to fit the proof to this modified statement, and in this latter form the theorem applies directly to cases to which the other form is not applicable.

BROMWICH'S THEOREM: *If the series Σu_n is summable $(C1)$ to the value S, and the set of functions, $f_0(\alpha), f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha), \cdots$, defined for a set of values $E(\alpha)$ having at least one limit point α_0, not of the set, satisfies the conditions*

* Bromwich's theorem has been generalized to cases where k is non-integral by Chapman and Ottolenghi. Cf. [3] and *Giornale di Matematichs di Battaglini*, vol. 49 (1911).

$$\left.\begin{array}{ll} \text{(A)} & \sum_{n=p}^{n=q} n|\Delta^2 f_n(\alpha)|^* < K \quad \left(\begin{array}{l} p, q \text{ any integers} \\ K \text{ a positive constant} \end{array}\right) \\ \text{(B)} & \lim_{n\to\infty} nf_n(\alpha) = 0 \\ \text{(C)} & \lim_{\alpha\to\alpha_0} f_n(\alpha) = 1, \end{array}\right\} E(\alpha),$$

then the series $\Sigma u_n f_n(\alpha)$ *will converge absolutely over* $E(\alpha)$ *and the function* $F(\alpha)$ *which it defines there will be such that* $\lim_{\alpha\to\alpha_0} F(\alpha) = S$.†

Since we have, using the notation (28) for the case $k = 1$,

$$u_n = s_n - s_{n-1} = (S_n - S_{n-1}) - (S_{n-1} - S_{n-2})$$
$$= S_n - 2S_{n-1} + S_{n-2},$$

we obtain, if we set $S_{-2} = S_{-1} = 0$,

$$\text{(30)} \quad \sum_{m=0}^{m=n} u_m f_m(\alpha) = \sum_{m=0}^{m=n} (S_m - 2S_{m-1} + S_{m-2}) f_m(\alpha)$$
$$= \sum_{m=0}^{m=n-1} S_m \Delta^2 f_m(\alpha) + S_n f_n(\alpha) - S_{n-1} f_{n+1}(\alpha).$$

In view of the hypothesis that Σu_n is summable $(C1)$ we may choose a positive constant C such that $|S_n| < (n+1)C$ for all values of n. Then, making use of this fact and condition (A), we see that the summation on the right hand side of (30) approaches a limit over $E(\alpha)$ as n becomes infinite. Making use of the property of S_n just employed and condition (B), it follows that the remaining two terms on the right hand side of (30) approach zero as n becomes infinite. Thus the left hand side of (30) approaches a limit also, and we have

$$\text{(31)} \qquad F(\alpha) = \sum_{m=0}^{\infty} u_m f_m(\alpha) = \sum_{m=0}^{\infty} S_m \Delta^2 f_m(\alpha).$$

Applying this identity to the series $1 + 0 + 0 + 0 + \cdots$, we obtain, since in this case $S_m = m + 1$,

$$\text{(32)} \qquad f_0(\alpha) = \sum_{m=0}^{\infty} (m+1)\Delta^2 f_m(\alpha).$$

* The notation $\Delta^2 f_n(\alpha)$ is used as an abbreviation for $f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)$.

† If the terms u_n are functions of a variable and Σu_n is uniformly summable throughout a certain interval, the limit of $F(\alpha)$ will be approached uniformly over that interval as α approaches α_0.

From (31) and (32) we obtain

$$(33)\quad F(\alpha) - Sf_0(\alpha) = \sum_{m=0}^{\infty}\left(\frac{S_m}{m+1} - S\right)(m+1)\Delta^2 f_m(\alpha).$$

Since Σu_n is summable $(C1)$ to S, we may choose an r such that

$$(34)\quad \left|\frac{S_m}{m+1} - S\right| < \frac{\epsilon}{4K} \qquad (m \geqq r),$$

ϵ being an arbitrary positive quantity and K the K of condition (A). Then, r being fixed, we may in view of condition (C) choose δ such that

$$(35)\quad \left|\sum_{m=0}^{m=r-1}\left(\frac{S_m}{m+1} - S\right)(m+1)\Delta^2 f_m(\alpha)\right| < \frac{\epsilon}{2}$$
$$(|\alpha - \alpha_0| < \delta).$$

From (33), (34), condition (A) and (35) we obtain

$$|F(\alpha) - Sf_0(\alpha)| < \epsilon \qquad (|\alpha - \alpha_0| < \delta).$$

Whence

$$\lim_{\alpha \to \alpha_0} F(\alpha) = \lim_{\alpha \to \alpha_0} Sf_0(\alpha) = S,$$

and our theorem is proved.

§3. *Applications to Problems in the Flow of Heat.*

The theorems about convergence factors are of special interest in view of the fact that they have important applications in connection with certain problems in mathematical physics. We will illustrate this by discussing a particular problem in the flow of heat.

We take the case of a finite rod of length π whose ends are maintained at zero temperature and whose surface is a non-conductor. Suppose the cross section of the bar is so small that the temperature is sensibly constant throughout it, and suppose the initial temperature is given by the function $f(x)$. We wish to determine the temperature of any point of the bar at any later moment.

Our problem reduces to the determination of a function of x and t, $v(x, t)$, such that

$$(\alpha)\quad \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad \begin{pmatrix} 0 < x < \pi \\ t > 0 \end{pmatrix},$$

$$(\beta) \quad v(0, t) = 0 = v(\pi, t) \qquad (t > 0),$$

$$(\gamma) \quad \lim_{\substack{t \to +0 \\ x \to x_1}} v(x, t) = f(x_1),$$

where x_1 is any point in whose neighborhood $f(x)$ is continuous. If we assume that $v(x, t)$ may be expressed in the form $u(x)w(t)$, we find as particular solutions of (α)

$$(36) \quad Ae^{-ka^2t} \sin ax, \quad Be^{-ka^2t} \cos ax \quad (A, B \text{ and } a \text{ constants}).$$

The second solution does not satisfy the boundary condition (β), so we reject it. The first solution satisfies (β), but will satisfy (γ) only in case $f(x) = \sin ax$. If $f(x)$ does not have this form, we naturally try to build up a sum of particular solutions of (α) of the form of the first solution in (36), $\Sigma A_n e^{-ka_n^2t} \sin a_n x$, which will approach $f(x)$ as t approaches $+0$. This raises the question of the possibility of expressing $f(x)$ in the form $\Sigma A_n \sin a_n x$. We know from the theory of convergent Fourier series that an arbitrary function of x, satisfying fairly wide conditions, may be expressed in a series of this form. But we also know that there exist continuous functions of x whose Fourier development diverges at points everywhere dense.

Our physical intuition tells us that there must be a solution of our problem corresponding to any original distribution of temperature that is thinkable, and therefore certainly in the case of an original distribution that is continuous. Fejér's theorem about summability $(C1)$ of the Fourier series, combined with some general theorem about convergence factors such as Bromwich's theorem, furnishes the mathematical demonstration that the series formed by introducing convergence factors of the type e^{-kn^2t} into the sine series for $f(x)$ is the desired solution.

The proof in detail is relatively brief. We have to show that the series

$$(37) \quad \sum_{n=1}^{\infty} A_n \sin nx e^{-kn^2t} \quad \left(A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \right)$$

converges for $(t > 0;\ 0 \leqq x \leqq \pi)$, and defines in that region a function $v(x, t)$ which satisfies conditions (α), (β) and (γ). We know from the Riemann-Lebesgue theorem referred to above that A_n approaches zero as a limit as n becomes infinite, and therefore the series (37) and the various derived

series obtained by differentiating (37) term by term with regard to t or x, will converge uniformly throughout the region $(t \geqq t_0 > 0, 0 \leqq x \leqq \pi)$. Thus we see that the series (37) defines a function $v(x, t)$ that satisfies conditions (α) and (β). It remains to show that this function also satisfies (γ).

It is readily seen that $v(x, t)$ will satisfy (γ) provided it approaches $f(x)$ as a limit as t approaches $+ 0$, uniformly throughout any interval included in an interval in which $f(x)$ is continuous. Since, from Fejér's theorem, $\Sigma A_n \sin nx$ is uniformly summable throughout any interval included in an interval of continuity of $f(x)$, it will follow from Bromwich's theorem that $v(x, t)$ approaches $f(x)$ uniformly throughout such an interval, in case the convergence factors in (37) satisfy the conditions of that theorem.

That conditions (B) and (C) are satisfied is easily seen. Turning to condition (A), we find that $\Delta^2 e^{-kn^2t}$ is negative when $n \leqq \sqrt{1/2kt} - 2$ and is positive when $n \geqq \sqrt{1/2kt}$. Hence this expression cannot change sign more than three times for any value of $t > 0$, and therefore condition (A) will be satisfied i we can determine a positive constant such that any sequence of terms chosen from $\Sigma n\Delta^2 e^{-kn^2t}$ is less in absolute value than this constant for all values of $t > 0$.

We have

$$\begin{aligned}\sum_{n=p}^{n=q} n\Delta^2 e^{-kn^2t} &= pe^{-kp^2t} - (p-1)e^{-k(p+1)^2t} \\ &\quad - (q+1)e^{-k(q+1)^2t} + qe^{-k(q+2)^2t} \\ &= p[e^{-kp^2t} - e^{-k(p+1)^2t}] - q[e^{-k(q+1)^2t} \\ &\quad - e^{-k(q+2)^2t}] + e^{-k(p+1)^2t} - e^{-k(q+1)^2t} \\ &= 2kp(p+\theta_1)te^{-k(p+\theta_1)^2t} - 2kq(q+\theta_2)te^{-k(q+\theta_2)^2t} \\ &\quad + e^{-k(p+1)^2t} - e^{-k(q+1)^2t} \quad \begin{pmatrix} 0 < \theta_1 < 1 \\ 1 < \theta_2 < 2 \end{pmatrix}.\end{aligned}$$

Each of the first two terms on the right hand side is less in absolute value than the expression $2kye^{-y}$ for some value of $y > 0$, and since this expression and also e^{-y} remain finite for all values of $y > 0$, it follows that condition (A) is satisfied by the convergence factors of (37). Hence the series (37) defines a function $v(x, t)$ which satisfies condition (γ), and as we saw previously that this same function satisfies (α) and (β), it follows that our physical problem is completely solved.

§4. *Summability of Other Developments.*

The developments in orthogonal functions of one variable that have been most extensively studied are, aside from the Fourier series, the developments in Sturm-Liouville functions and in Legendre's and Bessel's functions. The Sturm-Liouville developments offer less difficulty and yield simpler results than the other two. This is due to the fact that the differential equations which define Legendre's and Bessel's functions have singular points in the interval in which we wish to develop an arbitrary function, while the differential equation for the Sturm-Liouville functions does not. It is in the neighborhood of these singular points that the developments in terms of the former functions are more complex in their behavior and are more difficult to handle.

Very complete results with regard to the summability ($C1$) of the Sturm-Liouville developments were obtained by Haar in his dissertation [11]. He showed in fact that the behavior of the development of any function having a Lebesgue integral was the same at any point as the behavior of the cosine development of the same function. This result follows readily from the following fundamental theorem:

HAAR'S THEOREM. *If we represent by $s_n(x)$ and $\sigma_n(x)$ the sums of the first $n+1$ terms of the Sturm-Liouville and cosine developments respectively of any function $f(x)$ having a Lebesgue integral, we have*

$$\lim_{n\to\infty} [s_n(x) - \sigma_n(x)] = 0$$

uniformly over the interval $(0 \leqq x \leqq \pi)$.*

This theorem enables us to infer from the various results obtained with regard to the summability of the Fourier's series, corresponding results for the Sturm-Liouville developments. Thus we obtain not only the analogue of Fejér's theorem, as pointed out by Haar, but also the analogues of Lebesgue's theorem, the Riesz-Chapman theorem, and Hardy's theorem.

The first study of the summability of the developments in Legendre's functions was made in 1908 by Fejér who established summability ($H2$), or what is equivalent ($C2$), at every

* If the interval of definition of the Sturm-Liouville functions is taken as some interval (a, b) differing from the interval $(0, \pi)$, we readily reduce that case to the above by a change of variable in the differential equation and boundary conditions by means of which the functions are defined.

point in the interval $(-1 \leqq x \leqq 1)$ at which the function is continuous or has a finite jump, provided the function is absolutely integrable [6]. In 1911 Haar proved the summability $(C1)$ of the development at interior points of the interval $(-1 \leqq x \leqq 1)$ for the case where the function developed is continuous throughout the whole interval [12]. During the same year Chapman independently established summability $(C, k > 1)$ for all points of the interval at which the function is continuous or has a finite jump, provided the function has a Lebesgue integral over the interval. He further obtained summability $(C, k > \frac{1}{2})$ at the end points and $(C, k > 0)$ at interior points for the case where the function satisfies certain additional restrictions as to the possession of limited variation [4]. In 1913 Gronwall obtained summability of the same orders with less restriction on the function developed. His requirement for the case of interior points is that $f(x)$ and $(1-x^2)^{(k/2)-(1/4)}f(x)$ should be absolutely integrable in $(-1, 1)$. For the end points he requires the absolute integrability of $f(x)$ and a further condition on the function in the neighborhood of the end point opposite to the one for which summability is established. This condition, in the case of summability at $+1$, demands that

$$\int_{-1}^{-1+\delta} f(x')\sigma_n\big((2n+1)P_n(x)P_n(x')\big)dx',$$

where $P_n(x)$ represents the Legendrian of the nth order and $\sigma_n^{(k)}(u_n) = S_n^{(k)}/A_n^{(k)}$, $S_n^{(k)}$ and $A_n^{(k)}$ being defined by equation (28), should, for some value of $\delta > 0$, approach zero as n becomes infinite [10].

In the case of the developments in Bessel's functions, summability $(C1)$ at points in the interval $(0 < x < 1)$ where the function developed is continuous or has a finite jump, was established by the writer in 1908 [15] for the case of a function that is finite and integrable. Summability $(C1)$ for points of the same nature in $(0 < x \leqq 1)$, was established by W. B. Ford in his recent book on divergent series and summability [7] for the case of a function $f(x)$ that becomes infinite at a finite number of points while $\sqrt{x}f(x)$ remains absolutely integrable. Summability $(C, k > 0)$ at points in the interval $(0 < x \leqq 1)$ at which the function is continuous or has a finite jump for a function $f(x)$ such that $\sqrt{x}f(x)$ has a Lebesgue

integral, and summability $(C1/2)$ for $x = 0$, provided $f(x)$ has a Lebesgue integral, is continuous at the origin and is such that $|(f(x) - f(0))/x^\gamma|$ is less than a positive constant for some value of $\gamma > 1/2$ and values of x in the neighborhood of zero, have been established by the writer in two papers that have been presented to the Society but not as yet published.* Thus we see that here as in the case of the developments in Legendre's functions, we have to put more restriction on the function developed at the point at which the differential equation corresponding to the functions in terms of which we develop has a singular point; moreover, even then we do not get summability of as low an order. The analogy between the two cases suggests the existence of a general theory† with regard to the summability of developments in orthogonal functions that satisfy linear differential equations with singular points, which will include the important features of both cases. As far as the writer is aware this general theory is yet to be developed.

§5. *The Summability of Double Series.*

All the different methods of summation used in connection with simple series can be readily generalized so as to furnish methods for summing double series. We shall discuss here only the extension of the simplest method, i. e., summation by arithmetic means of the sums, to double series. We consider the double series Σa_{ij}, and we set

$$s_{mn} = \sum_{i=0,j=0}^{i=m,j=n} a_{ij}, \quad S_{mn} = \sum_{i=0,j=0}^{i=m,j=n} s_{ij},$$

$$\sigma_{mn} = S_{mn}/(m+1)(n+1).$$

Then if $\lim_{m,n\to\infty} \sigma_{mn}$ exists and is equal to σ, we say that the series Σa_{ij} is summable $(C1)$ to the value σ. It is easy to establish the consistency of this definition with the Pringsheim definition of convergence, for series such that $|s_{mn}| < C$, a positive constant, for all values of m and n. Moreover, in that case it will also be true that $|\sigma_{mn}| < C$ for all values of m and n.

The theorem of Frobenius for simple series has been extended to double series by Bromwich and Hardy [2], and Brom-

* For abstracts of these papers, see BULLETIN, vol. 24, pp. 65 and 422.

† Cf. a principle of generalization laid down by E. H. Moore, Introduction to a Form of General Analysis, p. 1.

wich's theorem for simple series has been extended to double series by the writer [16]. Fejér's theorem about the summability of the ordinary Fourier series has been extended to the double Fourier series, for points of continuity* of the function developed, by W. H. Young [21] and the writer [16], and for points of discontinuity of certain types by the writer [17]. These extensions, taken in connection with the extension of the theorem on convergence factors, enable us to discuss certain problems in mathematical physics in which double Fourier series occur, in a manner analogous to the discussion of the problem in the flow of heat considered earlier in the paper. Such a discussion may be found in [16].

The consideration of the summability of the double Fourier series naturally suggests the consideration of the summability of double series involving other orthogonal functions. Furthermore, the extension of the conception of summability to double series leads naturally to its further extension to triple series and to multiple series of any order. In particular we might study the summability of the triple Fourier series and thus obtain results having important applications to mathematical physics, analogous to those mentioned in connection with the ordinary and double Fourier series. Many other special studies readily suggest themselves. Thus we see that the work already done on the summability of developments in orthogonal functions is only a beginning, and that a large unexplored field remains to be investigated.

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MODULAR SYSTEMS.

The Algebraic Theory of Modular Systems. By F. S. MACAULAY. [Cambridge Tracts in Mathematics and Mathematical Physics, No. 19.] Cambridge University Press, 1916. xiv + 112 pp.

A MODULAR system is an infinite aggregate of polynomials in n variables $x_1, x_2, \cdots, x_n$, defined by the property that if F, F_1, F_2 belong to the system, $F_1 + F_2$ and AF also belong to the system, where A is any polynomial in $x_1, x_2, \cdots, x_n$. Hence if $F_1, F_2, \cdots, F_k$ belong to a modular system so also does $A_1F_1 + A_2F_2 + \cdots + A_kF_k$, where $A_1, A_2, \cdots, A_k$ are arbitrary polynomials in $x_1, x_2, \cdots, x_n$. In the algebraic theory (to which this tract is devoted) polynomials such as F and aF, where a is a quantity not involving the variables, are regarded as the same polynomial.

Let us consider any given infinite aggregate of polynomials in $x_1, x_2, \cdots, x_n$. Of those of any given degree L there are only a finite number which are linearly independent. Hence we may select an ordered sequence of polynomials $F_1, F_2, F_3, \cdots$ of the given aggregate such that any given polynomial of the aggregate is linearly expressible in terms of a finite number of the polynomials $F_1, F_2, F_3, \cdots$. Concerning such an ordered enumerable sequence of polynomials Hilbert has established the following theorem:

If $F_1, F_2, F_3, \cdots$ is an infinite sequence of polynomials in n variables $x_1, x_2, \cdots, x_n$, then there exists a finite number k such that for $h > k$ we have a relation of the form

$$F_h = A_{h1}F_1 + A_{h2}F_2 + \cdots + A_{hk}F_k$$

in which the A_{ij} denote polynomials in $x_1, x_2, \cdots, x_n$.

Our author gives on page 38 essentially König's proof of this theorem. Since the proof is from first principles, he is justified in assuming the theorem from the beginning.

From the foregoing considerations it follows that every infinite aggregate of polynomials is contained in the infinite aggregate of a modular system and that every modular system is itself the aggregate of polynomials

$$A_1F_1 + A_2F_2 + \cdots + A_kF_k$$

in which $F_1, F_2, \cdots, F_k$ are k properly chosen polynomials of the system and $A_1, A_2, \cdots, A_k$ are arbitrary polynomials. The polynomials $F_1, F_2, \cdots, F_k$ are said to form a basis of the system.

In its simpler aspects the theory of modular systems is of importance in geometry. Let it be required, for example, to determine the class of algebraic plane curves which pass through the intersection points of the curves

$$F_1(x, y) = 0, \quad F_2(x, y) = 0.$$

It is obvious that the class includes all the curves

$$A_1F_1 + A_2F_2 = 0$$

where A_1 and A_2 are polynomials in x and y. The question arises as to whether or under what circumstances every algebraic curve passing through the given intersection points can be represented by an equation of the last form above. The

answer to such a question affords one of the simplest applications of the algebraic theory of modular systems.

The primary object of the algebraic theory of modular systems is to discover those general properties of a system which will afford a means of answering the question whether a given polynomial is or is not a member of a given system. This calls for a generalization of the theory of the solution of equations in several unknowns. In order that a polynomial F may belong to a modular system with a given basis $F_1, F_2, \cdots, F_k$ it is obviously necessary that F shall vanish for all finite solutions of the system $F_1 = F_2 = \cdots = F_k = 0$. This condition is sufficient only if the given modular system has a certain property; otherwise, it is not sufficient, and F must satisfy further conditions, also connected with the solutions of the system $F_1 = F_2 = \cdots = F_k = 0$; and these may be difficult to express concretely.

At present the theory of modular systems is incomplete and offers a wide field for research. The subject is one of great difficulty, and he who works in it must exercise extreme care to avoid the pitfalls. Macaulay points out several mistakes made by his predecessors, some of them occurring in the most important memoirs dealing with the subject.

From what has been said it is evident that the first step in the theory of modular systems is to find all the solutions of the algebraic equations $F_1 = F_2 = \cdots = F_k = 0$. This is completely accomplished in the theories of the resultant and the resolvent, developed by our author in sections I and II (pages 3–28). The theory of the resultant of two homogeneous polynomials in two variables is first treated. There is given then a parallel development of the much more difficult theory of the resultant of n homogeneous polynomials in n variables. From this the necessary theory for non-homogeneous polynomials follows at once. Application is made to the solution of certain classes of algebraic equations. In the theory of the resolvent the author follows in the main König's exposition of Kronecker's method of solving equations by means of the resolvent.

General properties of modular systems are developed in section III (pages 29–63) The treatment is closely allied to Lasker's memoir (*Mathematische Annalen*, volume 60, 1905) and Dedekind's theory of ideals. Section IV (pages 64–100) contains an extension of Lasker's results founded on the methods

originated by Noether. The contents of this chapter are due to the author himself. A considerable number of the properties proved in the section have been established by him in previous memoirs. But a new method, that of the inverse system, is here employed for the first time and the results are closely associated with it. The author's own account of the method is to be found on pages 64 ff.

The monograph ends with a note of twelve closely printed pages containing a brief explanation of the theory of ideals of algebraic numbers and functions and of the relation in which the algebraic theory of modular systems stands with respect to it.

Throughout the tract the exposition is given in condensed form, evidently best adapted to the needs of investigators in the field. But a portion of the treatment, especially that of the first two sections, is suited to the needs of the general mathematical reader interested in the general aspects of the theory of algebraic equations in several unknown quantities.

R. D. CARMICHAEL.

NOTES.

THE regular meeting of the Chicago Section of the American Mathematical Society at the University of Chicago on Friday and Saturday, April 4–5, 1919, will include a symposium on the geometry of numbers with applications to questions of minima and algebraic numbers. Formal papers, based largely on the work of Minkowski, will be presented by Professor H. F. BLICHFELDT, of Stanford University, and Professor L. E. DICKSON, of the University of Chicago. Synopses of these papers will be sent out with the programmes of the meeting.

THE programme of the regular meeting of the Society in New York City on April 26, 1919, will include reports of the work of members of the Society in the government Ordnance Department at Washington and Aberdeen.

THE opening (January) number of volume 20 of the *Transactions of the American Mathematical Society* contains the following papers: "Necessary conditions in the problems of

Mayer in the calculus of variations," by GILLIE A. LAREW; "Linear equations with unsymmetric systems of coefficients," by ANNA J. PELL; "On convex functions," by HENRY BLUMBERG; "Projective transformations in function space," by L. L. DINES; "On the order of primitive groups (IV)," by W. A. MANNING.

There is a portrait of the late Professor MAXIME BÔCHER as frontispiece, with a short obituary note. Copies of this portrait may be obtained by sending twenty cents in postage stamps to the office of the Society, 501 West 116th Street, New York City.

THE opening (January) number of volume 41 of the *American Journal of Mathematics* contains the following papers: "Groups generated by two operators whose relative transforms are equal to each other," by G. A. MILLER; "A classification of general (2, 3) point correspondences between two planes," by T. R. HOLLCROFT; "The classification of plane involutions of order (3)," by ANNA M. HOWE; "On surfaces containing a system of cubics that do not constitute a pencil," by C. H. SISAM; "An isoperimetric problem with variable end-points," by A. S. MERRILL.

WITH the January, 1919, issue, the *Tôkyô Sûgaku-Buturigakkwai Kizi* (*Proceedings of the Tokyo mathematico-physical Society*) begins its third series. The name of the periodical changes with the new series to *Nippon Sûgaku-Buturigakkwai Kizi* (*Proceedings of the physico-mathematical Society of Japan*).

THE January number of the *American Mathematical Monthly* contains the names and rank of one hundred and seventy-eight teachers of mathematics in the national war service.

AT the recent annual meeting of the Mathematical Association of America Professor H. E. SLAUGHT was elected president and Professors R. G. D. RICHARDSON and H. L. RIETZ vice-presidents.

AT the meeting of the London mathematical society held December 12, the following papers were read: By G. H. HARDY and J. E. LITTLEWOOD, "Applications of the method of Faray dissection in the analytic theory of numbers: (1) A new solu-

tion of Waring's problem; (2) Proof that every large number is the sum of at most thirty-three biquadrates; (3) The Riemann hypothesis and the expression of a number as the sum of a stated number of primes"; by N. M. SHAH and B. M. WILSON, "Numerical data connected with Goldbach's theorem"; by M. FRÉCHET, "Integrals in abstract fields."

AT the meeting of the Edinburgh mathematical society on January 10, the following papers were read: By E. M. HORSBURGH, "A mechanical solution of a differential equation of torsion"; by F. BOWMAN, "Note on the intersection of a plane curve and its Hessian at a multiple point" and "Note on the formula for the radius of the circumsphere of a tetrahedron in terms of the edges."

THE Prince Jablonowski Society of Leipzig announces the following prize problem for 1921: "To extend the theory of linear functional differential equations in any direction. A complete treatment of new special cases is especially desirable." Competing memoirs should be submitted not later than October 31, 1920. The value of the prize is 1500 marks.

THE philosophical faculty of the University of Berlin announces the following prize problem: "To determine, by means of the theory of elementary divisors, the criteria that a given matrix be capable of representation as the composition of two skew-symmetric matrices." Competing memoirs should be presented before June 4, 1919.

THE academy of sciences of Heidelberg has joined the other learned societies as contributing member to the support of the Encyklopädie der mathematischen Wissenschaften.

COLLÈGE DE FRANCE. THE following courses in mathematics are announced for the session beginning December 2, 1918: By Professor G. HUMBERT: Theory of quadratic numbers, two hours.—By Professor J. HADAMARD: Influence of the form of the domain in the problems of mathematical physics, two hours.—By Professor M. BRILLOUIN: English and American theories of the gravitational stability of the earth, two hours.—By Professor L. LANGEVIN: The principle of relativity and the theories of gravitation, two hours.—By Pro-

fessor LE ROY: The present state of mathematical philosophy, and its relations with the philosophy of intuition, two hours.

PROFESSOR M. FRÉCHET, of the Sorbonne, has been appointed professor of mathematics in the reconstituted University of Strassbourg. During the summer semester of the present year (April 15–July 30) he will lecture on General Analysis (Cacul fonctionnel). The lectures will be in French, but Professor Fréchet offers to discuss them in English with any English-speaking students who wish to avail themselves of the opportunity.

THE University of Frankfort has conferred an honorary doctorate on Professor L. KÖNIGSBERGER, of the University of Heidelberg.

DR. V. GEILEN has been appointed docent in mathematics at the University of Münster.

PROFESSOR OTTO STAUDE, of the University of Rostock, has been chosen rector for the year 1918–19.

DR. V. GRILLIER, of the University of Biel, has been appointed professor of mathematics at the University of Bern.

PROFESSOR M. NOETHER, of the University of Erlangen, has retired from active teaching, after forty-six years of service.

THE Royal Society of London has awarded its Copley medal to Professor H. A. LORENTZ, of the University of Leyden, for his work in mathematical physics.

PROFESSOR W. FOORD-KELCEY, of the Royal military academy, has been appointed Officer of the Order of the British Empire, for services in connection with the war.

THE Rumford Committee of the American Academy of Arts and Sciences has voted the sum of $500.00 to Professor A. G. WEBSTER, of Clark University, in aid of his researches in pyrodynamics and practical interior ballistics.

THE committee recently organized to establish a suitable memorial of the late Professor MAXIME BÔCHER requests that

subscriptions to the memorial fund be sent to the treasurer of the committee, Professor J. H. TANNER, Cornell University, Ithaca, N. Y. Contributors are invited to express their views and wishes in regard to the use which should be made of the fund.

AT the University of Manitoba, Dr. H. R. KINGSTON has been promoted to an assistant professorship of mathematics. Assistant professor L. A. H. WARREN has been appointed acting professor of mathematics and astronomy, in the absence of Professor N. B. MACLEAN, who is major in the Canadian artillery service in France.

PROFESSOR E. R. HEDRICK, of the University of Missouri, has recently sailed for France, where he is to take charge of the mathematical work for American soldiers in the Y. M. C. A. system.

DR. L. R. FORD and Mr. R. S. TUCKER have been appointed instructors in mathematics at Harvard University.

AT Cornell University, Mr. P. A. FRALEIGH, who was stationed at Aberdeen proving ground, has returned to his instructorship in mathematics. Mr. H. L. SMITH, who was under Major F. R. MOULTON in Washington, and Mrs. HELEN B. OWENS have been appointed instructors in mathematics.

MR. L. E. ARMSTRONG, of Stevens Institute of Technology, has been promoted to an assistant professorship of mathematics.

MR. J. W. BALDWIN, of Michigan State Normal College, has been appointed instructor in mathematics in Detroit Junior College.

PROFESSOR D. A. ROTHROCK, of Indiana University, was recently elected a member of the State Legislature. He is relieved of duties at the University during the legislative session to begin in January.

MR. F. S. NOWLAN, of Bowdoin College, has been promoted to an assistant professorship of mathematics.

At Brown University, Mr. T. A. Cornell and Mr. W. R. Burwell have been appointed instructors and Dr. A. B. Frizell lecturer in mathematics. Mr. C. R. Adams has resigned his instructorship to accept the Grand Army fellowship.

Dr. Tobias Dantzig and Dr. G. A. Pfeiffer have been appointed instructors in mathematics at Columbia University.

At the University of Minnesota, Dr. C. H. Yeaton has been appointed instructor in mathematics. Professor G. N. Bauer is on leave of absence, and Professor H. L. Slobin has resigned.

Mr. R. M. Mathews, of the Junior College, Riverside, Cal., has resigned to enter secondary teaching.

Mr. Arthur Ramsey, of Grove City College, has been promoted to an assistant professorship of mathematics.

At Northwestern University, the following new instructors in mathematics have been appointed: Miss Jessica M. Young, Mr. T. Doll, Dr. M. G. Smith, and Mr. P. E. Hemke.

At Wesleyan University, Mr. C. L. Stearns has been appointed instructor in mathematics and assistant in the Van Vleck observatory.

Professor G. Milhaud, of the University of Paris, well known for his writings on the history and philosophy of Greek mathematics, died October 1, 1918, at the age of sixty years.

Professor F. Daniels, of the University of Fribourg, Switzerland, died November 16, 1918, at the age of fifty-eight years.

Professor Ulisse Dini, of the R. Scuole Normale Superiore of Pisa, and editor of the *Annali di Matematica*, died October 28, 1918, at the age of seventy-three years.

Professor L. M. Sylow, of the University of Christiania, died September 7, 1918, at the age of eighty-five years.

PROFESSOR E. LAMPE, of the technical school at Berlin, died September 4, 1918, at the age of seventy-eight years. He was an editor of the *Jahrbuch über die Fortschritte der Mathematik* and of the *Archiv der Mathematik und Physik*.

PROFESSOR P. VON SCHAEWEN died April 28, 1918, at the age of seventy-one years.

The death is announced of Commander A. N. SKINNER, U. S. N., retired, professor of mathematics at the U. S. Naval Academy from 1898 to 1907.

DR. PAUL CARUS, for many years editor of the *Monist* and the *Open Court*, died at La Salle, Ill., on February 11, 1919.

DR. G. M. GREEN, of Harvard University, author of numerous memoirs in projective differential geometry, died January 24, 1919, at the age of twenty-seven years. He had been a member of the American Mathematical Society since 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIEBERBACH (L.). Differentialrechnung. Leitfäden für den mathematischen und technischen Hochschulunterricht. Leipzig, Teubner, 1917. M. 2.80

BRENDEL (M.). See GALLE (A.).

CARATHÉODORY (C.). Vorlesungen über reelle Funktionen. Leipzig, Teubner, 1918. M. 30.00

DIECK (W.). Nichteuklidische Geometrie in der Kugelebene. (Mathematisch-physikalische Bibliothek, Band 31.) Leipzig, Teubner, 1918. M. 1.00

FRICK (H.). Ueber den Zusammenhang der Perioden quadratischer Formen positiver Determinante mit der Zerlegung einer Zahl in die Summe zweier Quadrate. (Diss., Eidgenöss. Technische Hochschule.) Zürich, 1918.

GALLE (A.) und STÄCKEL (P.). Materialien für eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein, M. Brendel, und L. Schlesinger. Heft 4: C. F. Gauss als Zahlenrechner von A. Galle; Heft 5: C. F. Gauss als Geometer von P. Stäckel. Leipzig, Teubner, 1918. Gr. 8vo. 142 pp. M. 5.60

KLATT (—.). See SCHMIEDEBERG (—.).

KLEIN (F.). See GALLE (A.).

LEVI (F.). Abelsche Gruppen mit abzählbaren Elementen. Habilitationsschrift. Leipzig, Teubner, 1918.

LUCKEY (P.). Einführung in die Nomographie. 1ter Teil: Die Funktionsleiter. (Mathematisch-physikalische Bibliothek, Band 28.) Leipzig, Teubner, 1918. M. 1.00

D'OVIDIO (E.). See SCRITTI MATEMATICI.

SCHLESINGER (L.). See GALLE (A.).

SCHMIEDEBERG (—.), WETZSTEIN (—.) und KLATT (—.). Die Bedeutung des mathematischen und naturwissenschaftlichen Unterrichts für die Erziehung unserer Jugend. Preisschriften des Vereins zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts. Berlin, Salle, 1917. M. 4.50

SCHNEIDER (R.). Die Beweisführung für die Richtigkeit des Fermatschen Satzes. Eine mathematische Studie. 2 Bände. Budapest, 1918. M.10.00 + 10.00

SCRITTI MATEMATICI offerti ad Enrico d'Ovidio in occasione del suo LXXV genetliaco 11 agosto 1918. Torino, Fratelli Bocca, 1918. 8vo. 16 + 386 pp. L. 30.00

SIMON (M.). Analytische Geometrie der Ebene. 3te Auflage. Berlin, 1918. M.1.00

SLICHTER (C. S.). Elementary mathematical analysis. 2d edition. New York, McGraw-Hill, 1918. 497 pp. $2.50

STÄCKEL (P.). See GALLE (A.).

SÜNDERHAUF (—.). Einführung in die höhere Mathematik. 2tes Heft: Determinanten. (Beihefte zur Zeitschrift Lehrerfortbildung.) Leipzig, Haase, 1918. 64 pp. M. 2.00

VEBLEN (O.) and YOUNG (J. W.). Projective geometry. Volume 2, by O. Veblen. Boston, Ginn, 1918. 8vo. 12 + 511 pp. $5.00

WETZSTEIN (—.). See SCHMIEDEBERG (—.).

WEYL (H.). Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis. Leipzig, 1918. M. 4.00

YOUNG (J. W.). See VEBLEN (O.).

II. ELEMENTARY MATHEMATICS.

ADLER (A.). Fünfstellige Logarithmen. Mit mehreren graphischen Rechentafeln und häufig vorkommenden Zahlwerten. Berlin, 1918. M. 1.00

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Unterstufe. Ausgabe B. Leipzig, Teubner, 1917. 328 pp. Geb. M. 3.60

CRANTZ (P.). Planimetrie. 2te Auflage. (Aus Natur und Geisteswelt, Band 340.) Leipzig, 1918. 117 pp. M. 1.50

GLASER (R.). Sammlung von Aufgaben aus der Stereometrie. (Sammlung Göschen, Nr. 779.) Leipzig, Göschen, 1917. 168 pp.

GÖTTING (E.). See BEHRENDSEN (O.).

JACOB (J.), SCHIFFNER (F.), und TRAVNIČEK (J.). Lehrbuch der Arithmetik und Geometrie für Gymnasien und Realgymnasien. Wien, Deuticke, 1917–1918. Arithmetik, von Jacob, 1ter Teil: Unterstufe, 4te Auflage, 1917, 156 pp.; 2ter Teil, Mittelstufe, 2te Auflage, 1917, 146 pp.; Schlüssel: Die Theorie in Frageform und die Resultate der Aufgaben, 2ter Teil, 1917, 82 pp. Geometrie von Schiffner und Travniček, Mittelstufe, 2te Auflage, 1918, 221 pp. Geb.
Kr. 3.20 + 3.00 + 3.60 + 4.00

MAENNCHEN (P.). Geheimnisse der Rechenkünstler. 2te Auflage. (Mathematisch-physikalische Bibliothek, Band 13.) Leipzig, Teubner, 1918. 50 pp. M. 1.00

PFAU (J.). Erste Einführung in die Dreiecksrechenlehre (Trigonometrie) auf Grund der natürlichen Winkelfolgen zum Schul- und Selbstunterricht. Wien, Pichler, 1917. Geh. Kr. 3.60

SCHIFFNER (F.). See JACOB (J.).

SPORER (B.). Niedere Analysis. 2te Auflage. Berlin, 1918. M. 1.00

TIMERDING (H. E.). Der goldene Schnitt. (Mathematisch-physikalische Bibliothek, Band 32.) Leipzig, Teubner, 1918. M. 1.00

TRAVNIČEK (J.). See JACOB (J.).

WUNSCH (H.). Unterhaltende Rechenstunden. Wien, Gerold, 1918. 112 pp. Kr. 2.00

III. APPLIED MATHEMATICS.

AHRENS (C.). See LAUENSTEIN (R.).

ANACKER (K.). Praxis des Flugzeugbaues. Ein Handbuch der Flugtechnik. 3 Bände. Band 1: Das Flugzeug und sein Aufbau. Berlin, 1917. M. 6.00

ASTRONOMISCHER Kalender für 1918. 37ter Jahrgang. Wien, Gerold, 1917. Kr. 6.20

BARUCH (A.). Die Grundlagen unserer Zeitrechnung. (Mathematisch-physikalische Bibliothek, Band 29.) Leipzig, Teubner, 1918. M. 1.00

BRICK (H.). Die Telegraphen- und Fernsprechtechnik in ihrer Entwickelung. (Aus Natur und Geisteswelt, Band 235.) Leipzig, Teubner, 1918. M. 1.50

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band 6: Geodäsie und Geophysik, und Astronomie. 1ter Teil, Abteilung B, Heft 4. Leipzig, Teubner, 1918. M. 6.40

EXNER (F. M.). Dynamische Meteorologie. Leipzig, Teubner, 1917. Gr.8vo. 9 + 308 pp. Geb. M. 16.50

FINSTERWALDER (S.). Alte und neue Hilfsmittel der Landesvermessung. Festrede. München, 1918. M. 1.00

HAUBER (W.). Die Grundlehren der Statik starrer Körper. 3ter Neudruck. Leipzig, 1918. M. 1.00

HEGNAUER (H.). Schulzeichnen auf Grund elementarer Perspektive. Leipzig, Teubner, 1918. 9 pp. + 18 Tafeln. M. 5.00

HENSELING (R.). Sternbüchlein für 1917. Mit einem Beitrag von H. H. Kritzinger. Stuttgart, Franckhsche Verlagshandlung, 1917. 8vo. 86 pp. Geh. M. 1.00

KOPPE (M.). Die Bahnen der beweglichen Gestirne im Jahre 1918. Eine Tafel nebst Erklärung. Berlin, Springer, 1918. Geh. M. 0.60

KRITZINGER (H. H.). See HENSELING (R.).

LAUENSTEIN (R.). Die Festigkeitslehre. 13te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1917. M. 5.40

——. Die Mechanik. 10te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1917. M. 5.40

——. Die graphische Statik. 13te Auflage. Bearbeitet von C. Ahrens. Leipzig, 1918. M. 5.40

LECORNU (L.). La mécanique. Les idées et les faits. Paris, Flammarion, 1918. 16mo. 304 pp. Fr. 4.75

ROHR (M. v.). Die optischen Instrumente. (Lupe, Mikroskop, Fernrohr, photographisches Objektiv und ihnen verwandte Instrumente.) 3te, vermehrte und verbesserte Auflage. Leipzig, 1918. 137 pp. M. 1.50

SILBERSTEIN (L.). Elements of the electromagnetic theory of light. London, Longmans, 1918. 16mo. 48 pp. 3s. 6d.

TURRIÈRE (E.). Sur le calcul des objectifs astronomiques de Fraunhofer. Paris, Service géographique de l'Armée, 1918. 8vo. 123 pp.

USENER (H.). Der Kreisel als Richtungsweiser. Seine Entwickelung, Theorie und Eigenschaften. München, 1917. M. 8.00

WEBER (E.). Der Weg zur Zeichenkunst. (Aus Natur und Geisteswelt, Band 430.) 2te Auflage. Leipzig, Teubner, 1918. 86 pp. M. 1.50

WOLFF (H.). Karte und Kroki. (Mathematisch-physikalische Bibliothek, Band 27.) Leipzig, Teubner, 1917. M. 1.00